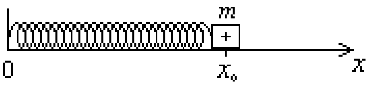
CHAPTER 11.

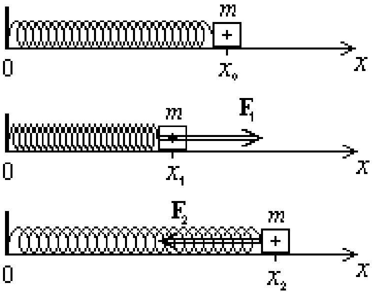
WAVES AND OSCILLATIONS

11.1 Springs

A helical spring should be a familiar object; like a length of rubber or a rubber band, springs have a particular physical property that make them especially suitable for physical analysis. When a spring is compressed or pulled (extended) the spring resists this deformation by trying to return to its original or natural length. The strength of this restoring force is proportional to the length by which the spring has been stretched, this property is known as Hooke's law. For the most simple discussion we will suppose that we have a very light spring that lies on a horizontal frictionless surface, we then suppose that a mass *m* is firmly fastened to one end of the spring while the other end is fixed to a side wall.



The centre of mass of *m* lies naturally at position x_0 and the spring has its natural length. When the spring is compressed *m* is moved to a new position x_1 and a force $\mathbf{F}_1$ from the spring attempts to push *m* back to its natural position, there is of course an equal and opposite push force between the spring and the wall at the other end of the spring.



When the spring is put under *tension* and is extended so that *m* is at x_2 a restoring force $\mathbf{F}_2$ attempts to pull the *m* back to its natural position, again there is an equal and opposite pull force between the spring and the wall. The equation for the forces are

$$\mathrm{F}_1 = -\mathrm{k}(x_1 - x_0)$$

or

$$\mathrm{F}_2 = -\mathrm{k}(x_2 - x_0).$$

If you consider this carefully you should observe that $x_1 < x_0$ so that F_1 is positive as it should be if it is to point along the x-axis; because $x_2 > x_0$, we also have a force F_2 that points in the negative x direction. The general equation for the force from a spring (that is fixed at one end) is

$$\mathrm{F} = -\mathrm{k}(x - x_0) \tag{11.1}$$

This force equation can also be written in terms of the mass *m* as

$$ma_x = -\mathrm{k}(x - x_0).$$

Now if we compare this equation with (3.14)

$$a \propto -(x - x_0)$$

which is the condition for harmonic motion, we can see that the motion of m on the end of the spring is meets this condition. Therefore we have a solution for the motion of the spring if we use equation (3.15)

$$x - x_0 = A\sin(\omega t + \phi).$$

If we again look back to the section on **HARMONIC MOTION** we will also find the equation

$$a_x = -\omega^2 r_x .$$

This can be compared with

$$a_x = -\frac{\mathrm{k}}{m}(x - x_0)$$

derived from equation (11.1). As r_x can be replaced by x - x_0 we can also identify

$$\omega^2 = \frac{\mathrm{k}}{m} . \qquad (11.2)$$

We have previously defined

$$\omega = \frac{2\pi}{\mathrm{T}}$$

where T was a period of rotation or a rotation projected along an axis. For our spring motion or harmonic motion, T is the period of a complete oscillation. For this repetitive motion we also have the relation between the period and the frequency f of oscillation

$$f = \frac{1}{T} . \qquad (11.3)$$

The motion of the mass attached the spring can be written as

$$x - x_0 = A\cos(\omega t) \qquad (11.4)$$

with the speed

$$v = \omega A\sin(\omega t) \qquad (11.5)$$

and the acceleration

$$a = -\omega^2 A\cos(\omega t). \qquad (11.6)$$

The angular frequency ω has been summarised above and relates to the mass and spring constant k by equation (11.2). The amplitude of the motion A is the maximum displacement from x_0, this can be either positive or negative.

When cos(ωt) = 1

$$x - x_0 = A$$
$$v = 0$$
$$a = -\omega^2 A .$$

The mass has reached a maximum distance from x_0 and come to a halt, the restoring force has also come to a maximum so the acceleration also has a maximum value, the mass has however come to rest before returning and the speed is zero.

When cos(ωt) = 0

$$x - x_0 = 0$$
$$v = \omega A$$
$$a = 0 \quad .$$

The mass is passing through the neutral or equilibrium position x_0 and is moving with a maximum speed but with no force is acting at this neutral position.

The kinetic energy of m is

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t)$$

with a maximum value of

$$(\tfrac{1}{2}mv^2)_{\max} = \tfrac{1}{2}m\omega^2 A^2 = \tfrac{1}{2}\mathrm{k}A^2 \, .$$

This is also the sum of the kinetic and potential energy of the system. If we write the potential energy stored in the spring as

$$P.E. = \tfrac{1}{2}\mathrm{k}(x - x_0)^2$$

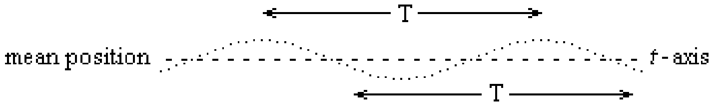
we can show that the total kinetic energy plus potential energy equals the energy in the system

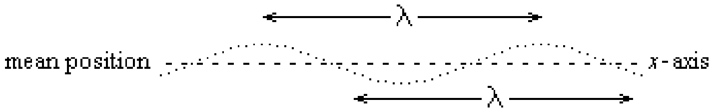
$$K.E. + P.E. = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\mathrm{k}(x - x_0)^2 = \tfrac{1}{2}\mathrm{k}A^2 \, . \qquad (11.7)$$

In our frictionless, conservative system we would expect this energy sum to remain constant.

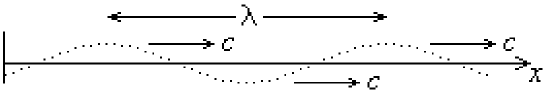
11.2 Movement in Waves

A wave can move across or through a medium when the particles (or parts) of the medium move harmonically. It is usual to assume that the particles of the medium simply oscillate about a mean position while the wave moves with a speed c. We need to distinguish the speed of the wave, c from the speed of a particle, v in the medium. As the oscillation spreads through space, wavelengths are defined, a wavelength, λ is the distance between a displacement and the nearest identical displacement at these points the medium will also have an identical motion.





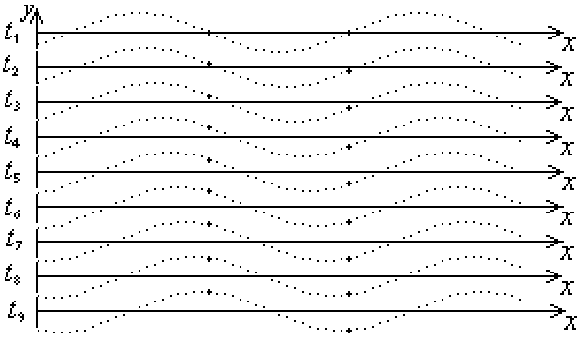
Our next formula relates the frequency, speed and the wavelength.



In the second diagram the speed of the wave is indicated. As the wave moves to the right one wavelength takes one period to pass so we can simply write

$$c = \frac{\lambda}{T} = \lambda f \qquad (11.8)$$

There are two types of wave that are characterised by the way that the particles of the medium move in relation to the motion of the wave. In a *transverse* wave the particles move at right angles to the direction that the wave travels. Water waves are an example of transverse waves.



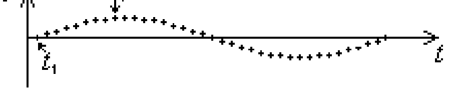
The above diagram shows a transverse wave moving through a line of particles with snapshots taken at equal time intervals; two particles have been tagged.

In a *longitudinal* wave the particles move in the same direction as the wave travels. Sound waves travelling through air molecules are an example of longitudinal waves.

The above diagram shows a longitudinal wave moving through a line of particles with snapshots taken at equal time intervals, the transverse displacements from the previous diagram have been drawn longitudinally. Again two tagged particles are shown. The background compressive waves can be seen moving to the right with a faster speed than that of the particles of the medium.

11.3 Moving Waves

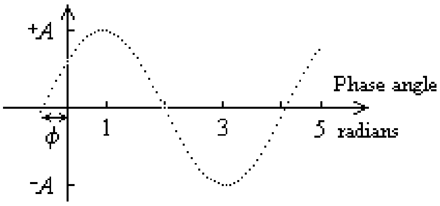
There is no obvious connection between the motion of a wave and the motion of particles in the medium through which the wave travels. In this section we want to concentrate on describing a travelling wave. If we replot our transverse wave against time we have the following result.



This wave takes the form

$$y = A\sin\left(\frac{2\pi}{T}t + \phi\right) = A\sin(\omega t + \phi)$$

where ϕ is an arbitrary angle (called a phase constant) and A is the amplitude of the motion.



The wave was also previously displayed along an x-axis and thus it may also take the corresponding form

$$y = A\sin\left(\frac{2\pi}{\lambda}x + \varphi\right) = A\sin(kx + \varphi)$$

where

$$k = \frac{2\pi}{\lambda} \quad . \tag{11.9}$$

We can combine these two forms of space and time waves to describe a travelling wave.

$$y = A\sin(\omega t - kx) \tag{11.10}$$

describes a wave travelling to the right along an x-axis, while

$$y = A\sin(\omega t + kx) \tag{11.11}$$

describes a wave travelling to the left along an x-axis.

11.4 The Standing Wave in a String

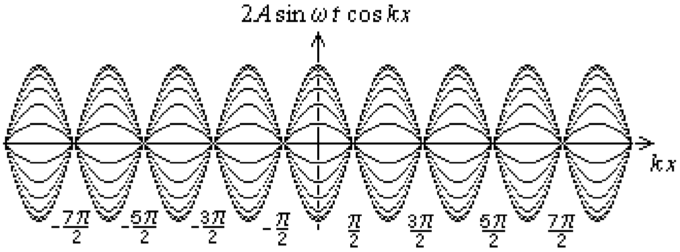
When a transverse wave is generated in a taut string of length l two travelling waves are formed, these have the same amplitudes only they travel in opposite directions. the displacements of overlapping waves simply add together and so the wave in the string has the form

$$y = A\sin(\omega t - kx) + A\sin(\omega t + kx)$$

$$y = A\sin\omega t\cos kx - A\cos\omega t\sin kx + A\sin\omega t\cos kx + A\cos\omega t\sin kx$$

$$y = 2A\sin\omega t\cos kx \tag{11.12}$$

This last equation is not that of a travelling wave, it describes the oscillation of a part of the string as having an amplitude of $2A\cos kx$ while the particle moves in time according to the factor sin ωt. A sketch of equation (11.12) is shown below.



The fixed ends of the string must correspond with a fixed zero in our function, these zero points are called *nodes*. The length of a number of half wavelengths of this standing wave must be equal to the length of the string; that is

where n is an integer. This equation gives us the wavelength in the string

$$\lambda_{\mathrm{n}} = \frac{2l}{\mathrm{n}} . \tag{11.13}$$

We have a discrete number of choices for the wavelength of a transverse wave in a string of length l, each particular value of n defines a *mode.* For a *fundamental* mode or *first harmonic* n = 1. For the *second harmonic* mode n = 2, for the *third harmonic* mode n =3 etc. As the speed of a wave in a string depends on the tension in the string we can also calculate the frequency of the modes (or harmonics) using equation (11.8)

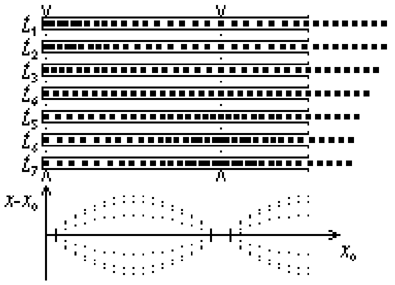
$$f_{\mathrm{n}} = \mathrm{n}\frac{v}{2l} . \tag{11.14}$$

If m is the mass of the string and F is the tension force, v can be found from

$$v = \sqrt{\frac{Fl}{m}} .$$

11.5 The Standing Wave in a Pipe

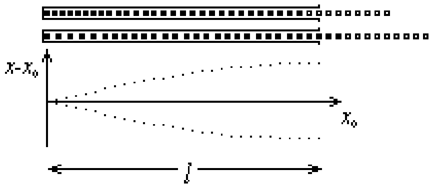
When a longitudinal sound wave is generated in a pipe of length l two travelling waves must again be formed. These have the same amplitudes only they travel in opposite directions and overlap or add together. At an open end of the pipe the air is fixed at the pressure of the outside atmosphere, while at a closed end the pressure can compress and rarefy against the end. From the point of view of the air molecules (particles in the medium) they are free to move into the atmosphere (displacement antinode) at an open end, but they are not free to move when confronted by the wall at an inner end (displacement node). The pressure antinode (or greatest variation of pressure) occurs at the displacement node, where the particles may be compressed. The pressure node (or zero change) corresponds to the displacement antinode where the particles can move freely without compression. We have tried to illustrate this point on the next page.



Here we have taken a single line of particles in a narrow, closed tube and displayed them at regular time intervals for half a period of their standing wave. We have indicated the lines where the displacement nodes occur; it should be obvious that at these are pressure antinodes. To maintain a pressure node at the open end of the tube particles are drawn in and out of the tube, outside the tube these particles stay at the same pressure (which is indicated by their regular spacing). If we were to write an equation for the position of the particles in the manner of equation (11.12) we could write:

$$x_{\mathrm{j}} = x_{\mathrm{j}0} + 2A\sin \omega t \cos kx$$

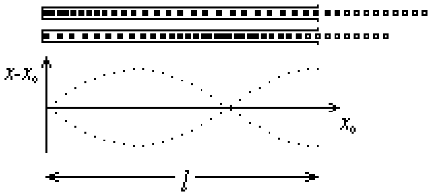
To find the standing waves in this open and closed pipe we use the knowledge that the closed end has a displacement node while the open end has an antinode. The fundamental mode or first harmonic looks like



so that

$$l = \frac{\lambda}{4}.$$

The second harmonic looks like



so that

$$l = \frac{3\lambda}{4}.$$

For the n^{th} harmonic we need

$$l = \frac{(2n-1)\,\lambda}{4}$$

so that

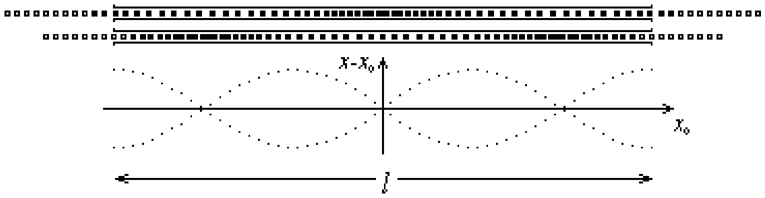
$$f_n = \frac{(2n-1)v}{4l}. \qquad (11.15)$$

If you are interested the speed of sound in air is found from the formula

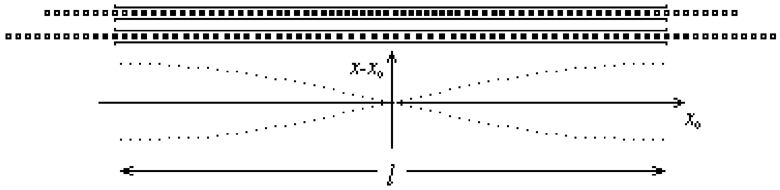
$$v = \sqrt{404(T+273)} \quad \mathrm{m\,s^{-1}},$$

where T is the air temperature on the Celsius scale.

If the pipe is open at both ends (this is unusual for musical instruments), displacement anti-nodes must occur at each end. A typical example is shown:



To find the standing waves in this open pipe we use the knowledge that both ends have antinodes. The fundamental mode or first harmonic looks like



so that

$$l = \frac{\lambda}{2}.$$

The second harmonic gives

$$l = \lambda.$$

The third harmonic is shown at the top of the page, this has

$$l = \frac{3\lambda}{2}$$

For the n^{th} harmonic we need

$$l = \frac{n\lambda}{2}$$

so that

$$f_n = \frac{nv}{2l}. \tag{11.16}$$