CHAPTER 4.

CENTRE OF MASS AND NEWTON'S LAWS

4.1 Mass and momentum

Mass is the property of a body that determines how that body reacts to forces, or in the case of gravitational force it determines how the body acts as a weight. To say this using a formula, mass is the m in $\mathbf{F} = m\mathbf{a}$ or $W = mg$, where $\mathbf{F}$ is force or W is weight.

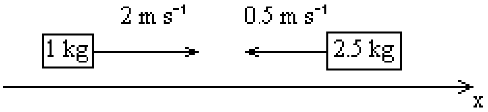
The units of mass are kilograms, kg.

Mechanically the response to a force shows in a change of motion, and/or a distortion. The motion of a body with mass can be described by its *momentum* which is found by taking the product of a body's mass with its velocity that is $m\mathbf{v}$**.** The units of momentum are kg m s^{-1}. The first important law states that: ***the momentum of a body cannot change unless a force from outside that body acts on it.*** This law is often called Newton's first law.

Under normal circumstances (in our earthly environment) frictional forces operate between bodies and the environment, these forces always tend to remove momentum. This constant loss of momentum is the reason why the above law was not generally understood until after the time of Gallileo and Newton. In the past the natural state of a body was thought to be at rest at the earth's surface, because ultimately all momentum was lost to friction unless driving forces were applied.

When two bodies collide their combined momentum on impact is conserved, even though friction may later bring these bodies to rest with their environment. Two (or more) bodies are called a *system* and the above law can be modified to say that: *the momentum of a system cannot change unless a force from outside the system acts on the system*.

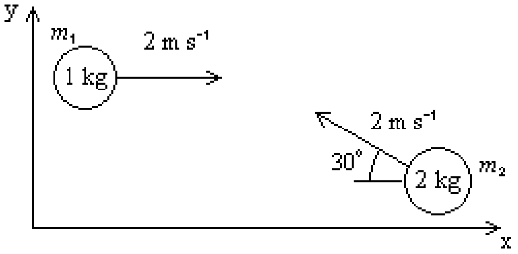
Example: Two masses of 1.0 kg and 2.5 kg approach each other with the speeds shown below, if they stick together after impact what is their final common speed?



Momentum in the x-direction before impact = $1\times2 + 2.5\times(-0.5) = 0.75$ kg m s^{-1}
Let the final speed be v, then the
momentum after impact of the combined bodies = $(1+2.5)v$
Momentum before = Momentum after
that is $0.75 = 3.5v$
so that $v = 0.21$ m s^{-1} .

As momentum is a vector quantity we need to apply our vector techniques for more problems in two or three dimensions.

Example: Two masses of m_1 = 1.0 kg and m_2 = 2.0 kg approach each other with the velocities shown in the following diagram. If the mass m_1 moves back along the x-axis with a speed of v_1' =1.0 m s^{-1} after impact, what is the final velocity of m_2?



To use vectors, we use the fact that conservation of momentum applies to both the x-component of the total momentum and the y-component of the total momentum. In mathematical terms we can write

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v_{1x}' + m_2 v_{2x}' \tag{4.1}$$

and

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v_{1y}' + m_2 v_{2y}' \tag{4.2}$$

where $\boldsymbol{v}_1'$ represents the velocity of m_1 and $\boldsymbol{v}_2'$ the velocity of m_2 after the collision. The problem tells us that $\boldsymbol{v}_1$ is 2.0 m s^{-1} in the x-direction, that is v_{1x} = 2.0 m s^{-1}. The problem also tell us that $\boldsymbol{v}_{1x}'$ is -1.0 m s^{-1} in the x-direction, but we are left to determine the components of $\boldsymbol{v}_2$.

$$v_{2x} = -2\cos 30° = 2\cos 150° = -1.73 \text{ m s}^{-1}$$

and $$v_{2y} = 2\sin 30° = 2\sin 150° = 1.00 \text{ m s}^{-1}.$$

Equation (4.1) gives $$1 \times 2 + 2 \times (-1.73) = 1 \times (-1) + 2v_{2x}'$$

or $$v_{2x}' = -0.23 \text{ m s}^{-1}.$$

Equation (4.2) gives $$1 \times 0 + 2 \times 1 = 1 \times 0 + 2 \times v_{2y}'$$

or $$v_{2y}' = 1.00 \text{ m s}^{-1}.$$

4.2 The centre of mass

When an external (or outside) force acts on an object (system or body) it can both change the momentum of the object and cause the body to rotate. As we wish to consider only the momentum changes, we require the forces to act so that they do not cause rotation of the body. For every body, there is a point at which all the mass of the body can be considered to act, this point is called the *centre of mass*. The centre of mass can be described as the mass averaged position of a body. When the line of action of a force passes through the centre of mass that force will not rotate the body.

If we consider the body as a collection of segments (the j^{th} mass is m_i), each centred at a location given by a position vector $\mathbf{r}_i$, then we can find the centre of mass using the following "mass average" formula

$$\mathbf{r}_{cm} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \,. \tag{4.3}$$

In a similar fashion we can also write the velocity of the centre of mass as

$$\mathbf{v}_{cm} = \frac{\sum_i m_i \mathbf{v}_i}{\sum_i m_i} \qquad (4.3a).$$

It is easiest to use these formulae by considering separate vector components.
Example: Consider three masses that make a 10 kg system with:

m_1 = 3.0 kg at x_1 = 2.0 m and y_1 = 3.0 m,
m_2 = 4.0 kg at x_2 = 4.0 m and y_2 = 1.0 m
and m_3 = 3.0 kg at x_1 = 1.0 m and y_1 = 2.0 m.

Find x_{cm} and y_{cm} the x- and y-positions of the centre of mass of the system.

To do this we can break up our vector equation into component equations as follows

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \qquad (4.3a)$$

and

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}. \qquad (4.3b)$$

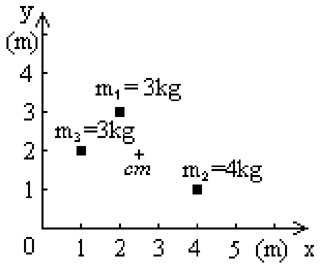
These equations enable us to write

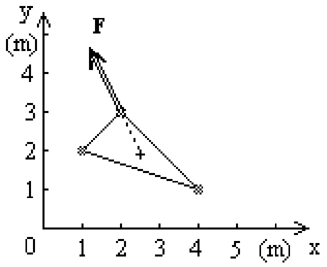
$$x_{cm} = \frac{3.2 + 4.4 + 3.1}{3 + 4 + 3} = \frac{25}{10} = 2.5 \text{ m}$$

and

$$y_{cm} = \frac{3.3 + 4.1 + 3.2}{3 + 4 + 3} = \frac{19}{10} = 1.9 \text{ m}.$$

If the three masses m_1, m_2 and m_3 were joined together by light rigid rods then any force that is in line with the centre of mass (shown by a cross) will not cause any rotation of the rigid system.





4.3 Motion caused by forces

The momentum of a body cannot be changed unless a force from outside that body acts on it. Our <u>second</u> law states: ***the change of momentum caused by a force is directly proportional to that force***. This law is also called Newton's second law. For a system with fixed mass (this is our usual assumption), any change in momentum can only involve a change in speeds or velocities and we can write the formula

$$F = \frac{\text{mass} \times \text{change of speed}}{\text{time for the change to occur}} = ma$$

or if we use vectors

$$\mathbf{F} = m\mathbf{a}. \qquad (4.4)$$

This should be a familiar (if not famous) formula and is quite straightforward to use. We usually assume that we have an external force that acts through the centre of mass of a body of mass *m* and this body is free to move in response to the force. The units of force are newtons, N;

$$1\ \text{N} = 1\ \text{kg m s}^{-2}.$$

We can also write the formula in a different form

$$F \times \textit{time for the change to occur} = \textit{change in momentum} \qquad (4.5)$$

The two formulae written in words can also be written using calculus, the first becomes

$$\mathbf{F} = \frac{\mathrm{d}(m\,\mathbf{v})}{\mathrm{d}t} \quad , \qquad (4.4a)$$

while the second is

$$\int_{t_1}^{t_2} \mathbf{F}\mathrm{d}t = m\mathbf{v}_2 - m\mathbf{v}_1. \qquad (4.5a)$$

Now as this course does not depend on calculus, we want an equivalent way of expressing these formulae. First we will overlook the fact that we are dealing with vectors, we can do this by simply keeping in mind that momentum and forces also have directions. Then we can make the following statements about the magnitude of the forces or momentum:

for equation (4.4a) (i) *the slope of the momentum against time graph gives the force that acts on a body*

and for equation (4.5a) (ii) *the area under the force against time graph gives the change of momentum.*

The last quantity *Force×time,* defined by equation (4.5) or (4.5a) has the particular name of *impulse.* When a body changes momentum abruptly, such as when a ball is struck with a bat (or racquet) we can describe the impulse as the product of the average force that acts on the ball while in contact with the bat, times the time during which the ball was in contact with the bat. This impulse is also a measure of the change in momentum of the ball.

Example: A tennis ball comes directly onto a racquet with a speed of 200 km hr^{-1} (56 m s^{-1}) and is returned with a speed of 150 km hr^{-1} (42 m s^{-1}). If the mass of the tennis ball is 0.055 kg and the ball is in contact with the racquet for 10 ms (this is 10 milli-seconds), what is the average force that the racquet exerts on the ball?

The change in speed of the ball = -42 - (56) = -98 m s^{-1}

The change in momentum of the ball = $m(v_f - v_i)$= 0.055(98) = 5.4 kg m s^{-1}

Average force

$$F = \frac{5.4}{10\times 10^{-3}} = 540\ \ \text{N}\ .$$

If this force was applied vertically it would lift about 54 kg against gravity.

4.4 Static equilibrium (the absence of relative motion)

There remains a third part to our laws of motion, or a third law sometimes called Newton's third law. This law states: ***every force is countered by an equal and opposite reaction***.

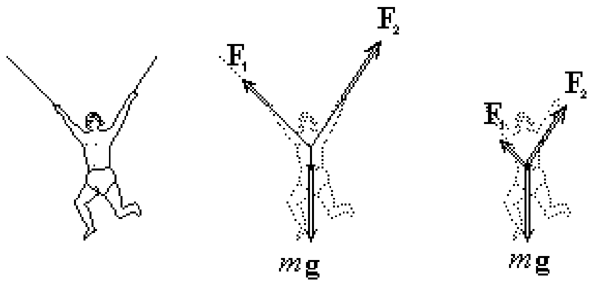
As we have seen some forces cause acceleration, these forces are opposed by the inertia of the mass on which they act, we call this an inertial reaction.

There is another class of forces, these just hold bodies together in their stable positions, without accelerating them. These are *equilibrium forces* as they keep objects from changing momentum. A *static equilibrium* is the condition where objects are at rest with respect to their surroundings. We can understand from experience what "at rest" means, but we should remember that our own rest frame of reference is moving all the time, that is we turn about the earths axis every day and we orbit the sun every year. We travel around the sun at about 30 000 m s^{-1} but as we all move together we experience no drastic forces or relative momentum changes in our environment.

The equilibrium forces are each countered by a balanced reaction according to our third law; in other words we can say that an equilibrium occurs when the sum of the forces acting on a body is zero. Mathematically we can write our equilibrium condition as

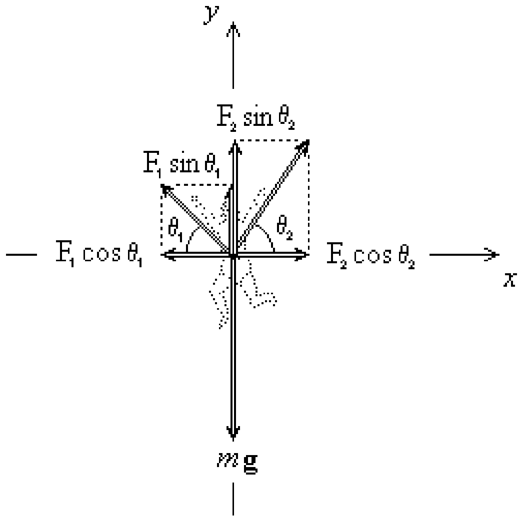
$$\sum \mathbf{F} = 0. \qquad (4.6)$$

Now to visualise the steps required to apply this formula, we will consider an exotic example shown below. While swinging from the vines in the jungle, Tarzan's attention was drawn to an extraordinary sight far below, so he stopped swinging and is hanging motionless from two vines.



Tarzan's weight, $m\mathbf{g}$ acts through his centre of mass (usually somewhere behind the navel for an upright person). His weight is supported by the two more forces, $\mathbf{F}_1$ and $\mathbf{F}_2$ that act along the vines from which he hangs. If Tarzan is to hang above the jungle without swinging or rotating then the forces acting on him must act from a common point; that is the lines of action of the three forces meet (somewhere in his chest). Now that we are concentrating on the forces, the unnecessary (but more interesting) details of Tarzan should be forgotten. In the last step the forces are drawn together all tail to tail as vectors. This last diagram with the supporting forces and the weight force all acting from the centre of mass is called a *free-body diagram.* As long as the external forces act from a common point the body is supported without relative acceleration or rotation. Because the total mass of the body can be regarded as acting at the centre of mass, we can also suppose that the external forces act on or from this point in the free-body diagram.

To find $\mathbf{F}_1$ and $\mathbf{F}_2$ we combine these with $m\mathbf{g}$ so that their sum is zero. This has been done in the following figure. The forces are still shown enlarged, Tarzan is now nearly forgotten, and a convenient choice of x and y axes has been made.



The original forces $\mathbf{F}_1$ and $\mathbf{F}_2$ need to be replaced with their x and y components

$$F_{1x} = F_1 \cos\theta_1 \text{ and } F_{1y} = F_1 \sin\theta_1$$

$$F_{2x} = F_2 \cos\theta_2 \text{ and } F_{2y} = F_2 \sin\theta_2 .$$

Equation (4.6) is
$$\sum \mathbf{F} = 0$$

and this can be broken into equations for vector components as

$$\sum_j F_{jx} = 0$$

and
$$\sum_j F_{jy} = 0 ,$$

for our example we can substitute to get

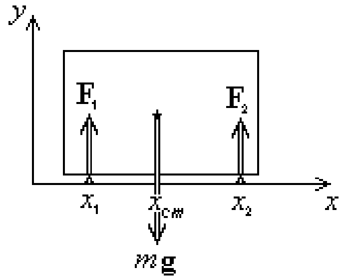
$$F_2 \cos\theta_2 - F_1 \cos\theta_1 = 0$$

and
$$F_2 \sin\theta_2 + F_1 \sin\theta_1 - mg = 0 .$$

It is these last two equations that will enable us to analyse the forces that hold Tarzan steady as he dangles high above the jungle floor.

4.5 Balance and moments

We continue with another aspect of equilibrium that ensures that a body does not rotate as it remains in equilibrium. Consider a simple mass supported by the two supports as shown below.



In this situation the support force $\mathbf{F}_1$ can only act through the support at x_1, while $\mathbf{F}_2$ can only act through the support at x_2. From the previous section (equation (4.6)) we know that

$$F_1 + F_2 - mg = 0 .$$

In this case the forces are all parallel and their lines of action cannot not meet at a point. We also know the weight of the block mg can be considered as acting from the centre of mass of the block so that the line of action of the weight force passes through x_{cm}. To counter the weight force we must suppose that the two forces act together through a "force averaged" position, x_s such a position would be

$$x_s = \frac{x_1 F_1 + x_2 F_2}{F_1 + F_2} .$$

Now, so that the weight can be supported, the force averaged position, x_s must be directly below the centre of mass

$$x_s = x_{cm} .$$

In this way, the weight force and the support forces are able to oppose (or balance) each other by acting in opposite directions along a common line of action. Combining these equations gives us

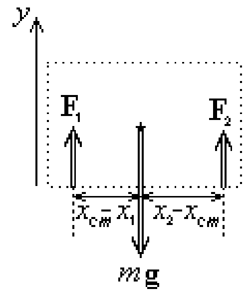
$$x_{cm} m g - x_1 F_1 - x_2 F_2 = 0 ,$$

this last equation is a statement about the *moments* of forces.

The moment of a force is the product of the magnitude of a force times the shortest distance from an axis to the line of action of the force; in our example the axis is the origin of the co-ordinate system. Our last equation illustrates the principle: ***for equilibrium the sum of the moments acting on a body must be zero***. When applying this principle a you may stop to ask "Where should the axis be?". The answer is simple, the axis may be at any convenient point. We can say, that for equilibrium: the sum of the moments about *any* axis must be zero. If we look back at the diagram we should note that the force m**g** acts to turn the block with a clockwise rotation about the axis at the origin, the corresponding moment $x_{cm}m$g is called a *clockwise moment*. The other two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ tend to support the system by reacting with anticlockwise rotation about the origin, their moments $x_1 F_1$ and $x_2 F_2$ are *anticlockwise moments*. In adding the moments together we must give the clockwise and anticlockwise moments different signs in their sum. This enables us to write the equation

$$\sum s_i F_i = 0 \qquad (4.7)$$

where s_i is the shortest (or perpendicular) distance from a common axis to the line of action of F_i and F_i has a sign that distinguishes forces that turn clockwise or anticlockwise about the axis.



In the last diagram the same forces as before are shown. This time an axis about (x_{cm}, 0) has been chosen, the shortest distances from this point to the lines of action of $\mathbf{F}_1$ and $\mathbf{F}_2$ are shown so that we can calculate the moments from the axis. The result is

$$\left(x_{cm} - x_1\right) F_1 - \left(x_2 - x_{cm}\right) F_2 = 0 ,$$

there is no turning moment for $m\mathbf{g}$ because it passes through the axis. If we are able to calculate the equilibrium condition from any axis then the last equation must also be true. You will find it is a combination of the previous equations

$$x_s = \frac{x_1 \mathrm{F}_1 + x_2 \mathrm{F}_2}{\mathrm{F}_1 + \mathrm{F}_2}$$

and

$$x_s = x_{cm}.$$

This theory of moments is essential for an understanding of the action of levers.