CHAPTER 3.

DESCRIBING MOTION

3.1 Speed and velocity

Speed is the rate at which a distance is covered. The *average speed* is the relative distance divided by the relative time.

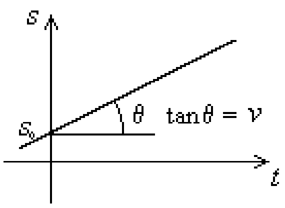
$$\bar{v} = \frac{s_1 - s_0}{t_1 - t_0} \qquad (3.1)$$

The *instantaneous speed* is defined for very small relative distances and times, however at this stage we will not worry too much about the distinction and we will discontinue the special bar notation that indicates the average of the speed.

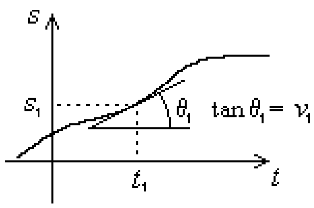
Our equation for speed can then be written in the form

$$s = vt + s_0, \qquad (3.2)$$

this is done by removing the subscripts " $_1$ " so that s_1 is represented by any general point s and t_1 becomes any general time *t*. We also put $t_0 = 0$, this is equivalent to starting our time measurement as we pass through the point s_0. Equation (3.2) is suitable for interpreting as a straight line graph.



If we have mastered our graphical analysis we will see that ***the speed is the slope of the graph of distance s, against time t***. This observation is generally true for all graphs of distance against time including those where the speed is changing so that the straight line becomes curved. The instantaneous speed of an object is the slope of the tangent of the distance against time graph at the particular instant of time, that is v_1 is the speed at t_1 in the graph below.



The average velocity is similarly defined using displacement vectors, that is

$$\bar{\mathbf{v}} = \frac{\mathbf{s}_2 - \mathbf{s}_1}{t_2 - t_1} = \frac{\mathbf{s} - \mathbf{s}_0}{t}, \qquad (3.3)$$

as before we need not continue with the bar notation to indicate an average. Although we are strictly describing speed against time graphs, these are frequently called "velocity time graphs". We can still use

$$\mathbf{s} = \mathbf{v}t + \mathbf{s}_0,$$

but we need to work more carefully when we are dealing with vectors.

The standard units for speed are metres per second (m s^{-1}) even though kilometres per hour are acceptable and in common use. We can convert between these systems of units by converting the units numerically, as an example:

$$27 \text{ km hr}^{-1} = 27 \text{ (1000 m) } (60 \times 60 \text{ s})^{-1}$$

or

$$27 \text{ km hr}^{-1} = 27000 \div 3600 \text{ m s}^{-1} = 7.5 \text{ m s}^{-1}.$$

As another example:

$$22.5 \text{ m s}^{-1} = 22.5\,(10^{-3}\,\text{km})\left(\frac{1\text{hr}}{60 \times 60}\right)^{-1} = 22.5 \times 10^{-3} \times 3600 \text{ km hr}^{-1}$$

or

$$22.5 \text{ m s}^{-1} = 81.0 \text{ km hr}^{-1}.$$

3.2 Acceleration

Just as speed is the rate of change of distance, so is acceleration the rate of change of speed.

$$\bar{a} = \frac{v - v_0}{t - t_0} = \frac{v - v_0}{t}. \qquad (3.4)$$

We can also define the acceleration as the rate of change of velocity, in this definition acceleration is also a vector.

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} = \frac{\mathbf{v} - \mathbf{v}_0}{t}. \qquad (3.5)$$

There are no distinct names for the scalar and vector forms of acceleration.

For constant acceleration the equation

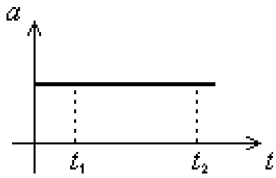
$$v = at + v_0 \qquad (3.6)$$

gives a straight line graph or more generally we can say that ***the acceleration is the slope of a speed against time graph***; when the acceleration is not constant the instantaneous acceleration is found from the slope of the tangent to the curve of v against t.

When the acceleration is constant, we have

$$v_2 - v_1 = a(t_2 - t_1).$$

It is easy to plot a constant acceleration



and the change in speed from the equation is seen to be the area under this acceleration against time graph. In calculus finding the area under a curve is also called integrating, this process enables us to find the change in speed from any acceleration against time graph. In general we can say: ***the change in speed between two particular times can be found from the area under the acceleration against time graph*** (even when this graph is neither constant or a straight line).

You should note the direct comparison between finding speed from the slope of a distance against time graph and finding the acceleration from the slope of a speed against time graph. These similar relations also enable us to state: ***the change in distance of a moving object can be found from the area under the corresponding speed against time graph*** (again this is generally true for all graphical forms).

3.3 Motion with constant acceleration

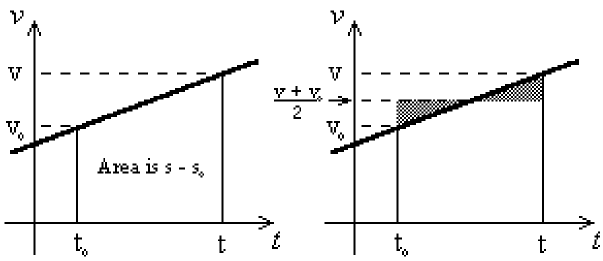
From equation ((3.2) we have

$$s - s_0 = vt$$

where we have assumed that v is constant (or there is no acceleration) and from equation (3.6) we have

$$v - v_0 = at$$

where we have assumed that a is constant. If we plot this last equation we also know that $s - s_0$ can be found from the area under the v against t graph.



Now in the diagram the average speed $\frac{v+v_0}{2}$ is exactly halfway between v and v_0; this enables us to cut the quadrilateral under $v = at + v_0$ and then rotate and slide the triangle as shown in the second part of the figure. In doing this symmetry operation we have not changed any areas so $s - s_0$ is also equal to the area of the new rectangle and we can write

$$s - s_0 = \frac{v + v_0}{2}\left(t - t_0\right)$$

or more generally

$$s - s_0 = \frac{v + v_0}{2} t \quad . \qquad (3.7)$$

We will repeat that this calculation for the area under a curve is only valid for a straight line graph, so equation (3.7) is valid when v changes uniformly, or the acceleration a is constant.

The three equations in this section can be rearranged to form other equations sometimes called *kinematic equations*, or even *equations of motion*. Equation (3.7) can be rearranged as

$$v + v_0 = \frac{2(s - s_0)}{t}$$

while we also have

$$v - v_0 = at$$

from equation (3.4), combining these (by multiplying common sides) gives the result

$$v^2 - v_0^2 = 2a(s - s_0) \quad . \qquad (3.8)$$

We can also substitute equation (3.6)

$$v = v_0 + at$$

into equation (3.7)

$$s - s_0 = \frac{v + v_0}{2} t$$

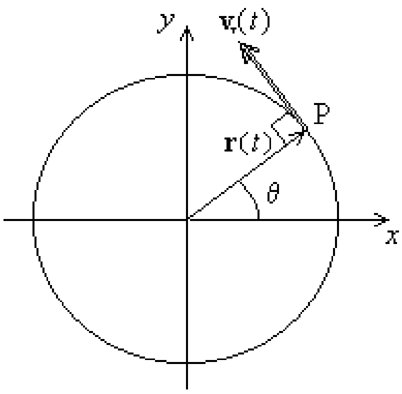
to get

$$s - s_0 = v_0 t + \tfrac{1}{2} a t^2 \quad . \qquad (3.9)$$

It is these last three equations that are commonly used as a set to describe motion of objects.

3.4 Rotation and oscillation

To describe rotational motion we will consider a point at P that rotates around the origin of the x-, y-axes in a circle about the origin. The point moves around the circle of radius r with a uniform (constant) speed v_T, the $_T$ indicates that the velocity always points tangentially to the circular path, the usual convention requires a positive rotation to be in the anti-clockwise direction.



The end of $\mathbf{r}(t)$ travels a distance $2\pi r$ in one "period" of time where T, the period is the time for one complete rotation. It should be obvious that the constant tangential speed is the distance around the circle divided by the time taken or

$$v = \frac{2\pi r}{\mathrm{T}} . \qquad (3.10)$$

To go once around the circle the angle θ changes by 2π radians and for any fraction of a period $\frac{t}{\mathrm{T}}$ we turn through an angle

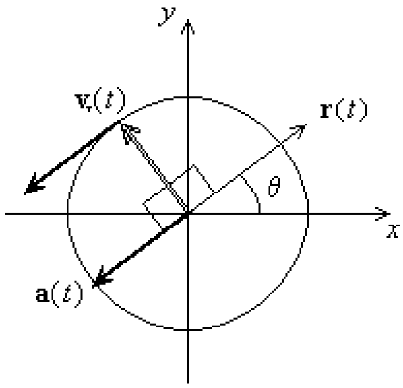
$$\theta = \frac{2\pi t}{\mathrm{T}} .$$

The quantity $\frac{2\pi}{\mathrm{T}}$ is known as the angular speed and has its own symbol ω so that

$$q = wt , \qquad (3.11)$$

and from equation (3.10) $v = wr$.

Having defined ω and the tangential speed $v_{Â}$ we can repeat this line of reasoning to find the acceleration $\mathbf{a}(t)$ that is required to change the tangential velocity as it also runs around a circle of radius r in one period T. This acceleration will not change the tangential speed but changes the direction of the tangential velocity. The change in velocity, or acceleration must be tangential, or perpendicular to $\mathbf{v}_{Â}(t)$ just as $\mathbf{v}_{Â}(t)$ is perpendicular to $\mathbf{r}(t)$.



The magnitude of the acceleration a is found from

$$a = \frac{2\pi v}{\mathrm{T}} = \omega v = \omega^2 r \qquad (3.12)$$

while the diagram shows that the direction of the acceleration $\mathbf{a}(t)$ is exactly opposite the direction of $\mathbf{r}(t)$. Further substitution using the above equations also gives the magnitude of the acceleration required to travel in a circular path with a constant speed as

$$a = \frac{v^2}{r} . \qquad (3.13)$$

3.5 Harmonic motion

If we return to the previous section, we started with a vector **r**(t) moving in a circle with a changing angle

$$\theta = \omega t$$

between **r**(t) and the x-axis. The x-component of this vector can be written as

$$r_x = r\cos\theta = r\cos(\omega t)$$

while the y-component can be written as

$$r_y = r\sin\theta = r\sin(\omega t).$$

The vector $\mathbf{v}_{\hat{A}}(t)$ is quarter a cycle or $\frac{\pi}{2}$ ahead of **r**(t) and its components can be written as

$$v_x = v\cos\left(\omega t + \tfrac{\pi}{2}\right) = v\left(\cos\omega t\cos\tfrac{\pi}{2} - \sin\omega t\sin\tfrac{\pi}{2}\right) = -\omega r\sin\omega t$$

and

$$v_y = v\sin\left(\omega t + \tfrac{\pi}{2}\right) = v\left(\sin\omega t\cos\tfrac{\pi}{2} + \cos\omega t\sin\tfrac{\pi}{2}\right) = \omega r\cos\omega t.$$

The vector **a**(t) is half a cycle ahead of **r**(t) or quarter of a cycle ahead of $\mathbf{v}_{\hat{A}}(t)$, the components can be written as

$$a_x = a\cos(\omega t + \pi) = a(\cos\omega t\cos\pi - \sin\omega t\sin\pi) = -\omega^2 r\cos\omega t$$

and

$$a_y = a\sin(\omega t + \pi) = a(\sin\omega t\cos\pi + \cos\omega t\sin\pi) = -\omega^2 r\sin\omega t.$$

By recollecting these equations we have

$$a_x = -\omega^2 r_x$$

and

$$a_y = -\omega^2 r_y$$

or more generally, by replacing r_x with $x - x_0$ and by replacing r_y with $y - y_0$ we have the equation for harmonic motion

$$\mathbf{a} \propto -(\mathbf{s} - \mathbf{s}_0). \qquad (3.14)$$

When this last condition, (3.14) applies in any single direction, we have the conditions for *Simple Harmonic Motion*, that is periodic motion that follows a sine curve with time. This motion is described by the equations

$$s - s_0 = A\sin(\omega t + \phi) \qquad (3.15)$$

$$v = \omega A\cos(\omega t + \phi)$$

and

$$a = -\omega^2 A\sin(\omega t + \phi).$$

In equation (3.15), s_0 is the centre point of the motion, A is called the amplitude, ω is the angular frequency (sometimes called the angular speed) and ϕ is the phase constant (in the previous section we used values of $\frac{\pi}{2}$ and π as phase constants).

For later consideration it is probably worth pointing out that if an object rotates with constant speed around a circle (at P at the end of **r**(t)) then any projection of this motion onto an axis or plane will follow harmonic motion. The shadow of an object on a round-about follows a sine curve (or of course a cosine curve) in time.

3.6 Numerical integration

Although this course does not require calculus it must (and has) demonstrate the principles that underlie this form of mathematics. We can start with the equation (3.6)

$$v = v_0 + at$$

and modify this back to

$$v = v_0 + a(t - t_0)$$

We can also go back to equation (3.7) and find

$$s - s_0 = \frac{v + v_0}{2}(t - t_0) = \bar{v}(t - t_0)$$

where $\bar{v}$ is the average speed and this particular definition is for a constant acceleration. If we apply our equations repeatedly for a continuous set of times t_0, t_1, t_2, t_3, ... etc., then we would have a set of equations

$$v_1 = v_0 + a(t_1 - t_0)$$
$$v_2 = v_1 + a(t_2 - t_1)$$
$$v_3 = v_2 + a(t_3 - t_2)$$
etc.

Now to mimic the approach of calculus, we make the essential assumption that these times are equally spaced and also very close together; that is $\Delta t = t_1 - t_0 = t_2 - t_1 = t_3 - t_2$ etc. so that

$$v_1 = v_0 + a\Delta t$$
$$v_2 = v_1 + a\Delta t$$
$$v_3 = v_2 + a\Delta t$$
etc .

As long as Δt is small we can also write

$$\bar{v} = \frac{v_1 + v_0}{2} \approx v_1 \approx v_0$$
$$\bar{v} = \frac{v_2 + v_1}{2} \approx v_2 \approx v_1$$
$$\bar{v} = \frac{v_3 + v_2}{2} \approx v_3 \approx v_2$$
etc .

so that our equations for s become

$$s_1 \approx s_0 + v_1\Delta t$$
$$s_2 \approx s_1 + v_2\Delta t$$
$$s_3 \approx s_2 + v_3\Delta t$$
etc .

Although we know that we are using approximations we will use a proper equals (" = ") sign with this last set. If we want to repeat these equations thousands of times our subscript values will grow and become unwieldy, in this case we replace the numbers by "running subscripts" j or j+1. These procedures leave us with two equations for iteration (or repeated use)

$$v_{j+1} = v_j + a\Delta t$$

and

$$s_{j+1} = s_j + v_{j+1}\Delta t.$$

The two iteration equations can easily be applied to a real life problem: if a ball is thrown straight up in the air with an initial speed of 20.0 m s^{-1}, how high does it rise?
The rate of downward acceleration caused by gravity is 9.80 m s^{-2}, or the acceleration a = -9.8 m s^{-2}. For this problem we have assumed that upwards is positive so that v_0 = 20.0 m s^{-1}, while Δt = 0.10 s will be small enough for our approximations to be reasonable. Finally we can assume the ball start at zero height, or s_0 = 0. Using the equations in sequence we will get

$$v_1 = 20 - 9.8 \times 0.1 = 19.02 \text{ m s}^{-1}$$

and $$s_1 = 0 + 19.02 \times 0.1 = 1.902 \text{ m}$$, after 0.1 s.

Repeating this procedure gives us

$$v_2 = 19.02 - 9.8 \times 0.1 = 18.04 \text{ m s}^{-1}$$

$$s_2 = 1.902 + 18.04 \times 0.1 = 3.706 \text{ m},$$

so that after a time interval of 2Δt = 0.20 s the ball has risen 3.71 m and has slowed to a speed of 18.0 m s^{-1}. This problem can best be continued by using spreadsheet or table:

$t_{j+1} = (j+1)\Delta t$	$v_{j+1} = v_j + a\Delta t$	$s_{j+1} = s_j + v_{j+1}\Delta t$
0	20	0
0.1	19.02	1.90
0.2	18.04	3.71
0.3	17.1	5.41
0.4	16.1	7.02
0.5	15.1	8.53
0.6	14.1	9.94
0.7	13.1	11.26
0.8	12.2	12.5
0.9	11.2	13.6
1.0	10.2	14.6
1.1	9.22	15.5
1.2	8.24	16.4
1.3	7.26	17.1
1.4	6.28	17.7
1.5	5.3	18.2
1.6	4.32	18.7
1.7	3.34	19.0
1.8	2.36	19.2
1.9	1.38	19.4
2.0	0.4	19.4
2.1	-0.58	19.4

At this point we have reached the time where the ball has stopped rising as the velocity has gone negative indicating that the ball is starting to fall down. This occurs at when t = 2.0 s and the height is s - s_0 = 19.4 m.

The equation $$v = v_0 + at$$

can be used with v = 0, v_0 = 20 m s^{-1} and a = -9.8 m s^{-2} to show that t = 2.04 s.

The equation $$s - s_0 = \frac{v + v_0}{2} t$$

can then be used to show that s - s_0 = 20.4 m. This is not the same value as the 19.4 m that is given from the table but this should not surprise us when we remember that the formulae were repeated approximations.