CHAPTER 2.

VECTORS

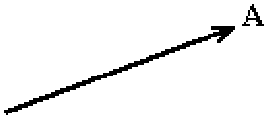
2.1 Distance and displacement

Distance is a measure for which you will not need and explanation; the standard measure of distance is the metre with kilometres and millimetres in constant use. We will return to a previous example to explain displacement, the statement "You will need to travel three kilometres to get home." is useful but you may also need to know the direction to travel. A distance with a direction is a *displacement*. Displacement is an example of a *vector,* a vector is a quantity with both a length (or magnitude) and a direction, for a vector quantity we will also need to identify the correct units for a complete description.

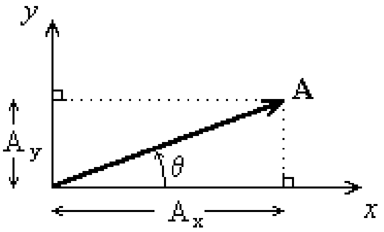
When we drive through the suburbs following a road map we drive along one street (in a particular direction for a particular distance) and then we turn and follow another street and so on. Each trip along a street is a displacement, our total journey is the sum of these displacements. Vectors add together in a similar fashion, one follows the other in sequence. If we take a short cut then another set of displacement vectors will get us to the same destination, if we travel directly in a straight line (as the crow flies) we find that one displacement vector represents the sum of all the other journeys.

2.2 Vectors

While displacement is a good example of a vector many other quantities are measured or specified in vector form, in this course we shall encounter forces, momentum, and the electric and magnetic fields. If we do not need to specify a direction we have a scalar quantity, distance is a scalar we may not need to know the direction we are going, the distance alone will help us estimate how long we will take and how best to travel. Other examples of scalars that we have met are mass and volume, time is another common scalar. Vectors are distinguished from scalar quantities using bold symbols thus, **A**. Other forms of notation are: $\underline{A}$ or $\vec{A}$ or $\overrightarrow{AB}$. A vector is represented in a diagram that indicates its magnitude by a (scaled) length and the direction by an arrow.



As things stand it is hard to describe this picture, this becomes easier if we add the usual *x* and *y* axes.



In this set of Cartesian (x,y) co-ordinates we have shown the perpendicular projections of **A** on to the x and y axes, these are called the *components* of the vector, the x-component is A_x and the y-component is A_y. As the projections form a rectangle (or two congruent right-angled triangles) we can find the length of the vector, A (often written |**A**|) using the Pythagorean theorem

$$A = \sqrt{A_x^2 + A_y^2} \quad .$$

You should also know enough trigonometry to recognise that

$$A_x = A\cos\theta$$

and $$A_y = A\sin\theta,$$

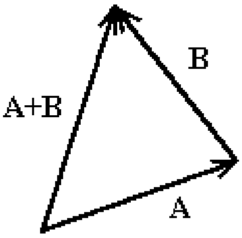
where θ is the anticlockwise angle to **A** from the x-axis. The slope of **A** is found from

$$\tan\theta = \frac{A_y}{A_x}$$

or $$\theta = \text{inv}\tan\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{A_y}{A_x}\right).$$

2.3 Adding vectors

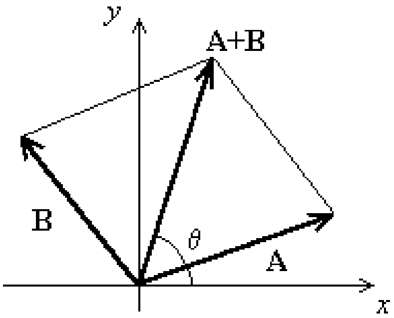
Vectors add together in the same fashion as the displacements added together, when we described a journey along different streets. One vector starts at the same point that the previous vector finished, the sum of the vectors is the direct displacement from the beginning of the first vector to the end of the last one.



While this picture is a useful representation of the vector sum we can only determine **A+B** using direct measurements with a ruler and a protractor. To determine the sum mathematically, we add the components

$$(A + B)_x = A_x + B_x \quad .$$

It is also a common practise to draw all vectors as starting at the origin, in this representation their sum is a diagonal of the parallelogram formed between the two vectors.



The length of **A+B** is

$$|\mathbf{A}+\mathbf{B}| = \sqrt{\left(A_x + B_x\right)^2 + \left(A_y + B_y\right)^2}$$

and the angle between **A+B** and the x-axis is found from

$$\tan\theta = \frac{\left(A_y + B_y\right)}{\left(A_x + B_x\right)}$$

In writing these formulae we had to use $|\mathbf{A}+\mathbf{B}|$ to describe the length of **A** + **B** the quantity described as A + B is not the length of **A** + **B**.

If you have already met diagrammatic vector representation, you will know that the difference of two vector requires a little thought before you can draw it. However if we use the algebraic form we can easily find the difference between two vectors

$$(A - B)_x = A_x - B_x$$

and

$$(A - B)_y = A_y - B_y.$$

2.4 Zero sums

When we start to discuss *force*s we will see that it is necessary condition for equilibrium is: *the sum of the forces acting on a body must be zero.* In common problems you will be given two forces and will need to find a third force that will give this equilibrium condition. The vector algebra that we must master to interpret equilibrium can be stated as: given two vectors find the third vector that gives them a zero sum, or given **A** and **B** find **C** so that:

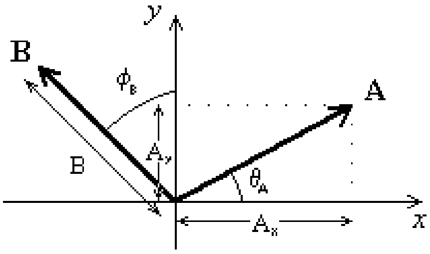
$$\mathbf{A} + \mathbf{B} + \mathbf{C} = 0.$$

The problem may well come in a mixed form, such as:

given $A_x = A\cos\theta_A$

$A_y = A\sin\theta_A$

and a vector of length B that makes an angle ϕ_B with the y-axis, what is the length and direction of a third vector that completes a zero sum?



To find **C** we need to know all of A_x, A_y, B_x and B_y. The angle from the x-axis to **B** is ϕ_B plus 90° and we can write

$$B_x = B\cos(\phi_B + 90°) = -B\sin\phi_B$$

and

$$B_y = B\sin(\phi_B + 90°) = B\cos\phi_B\,.$$

Next we rearrange

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$$

as

$$C_x = -A_x - B_x$$

and

$$C_y = -A_y - B_y\,,$$

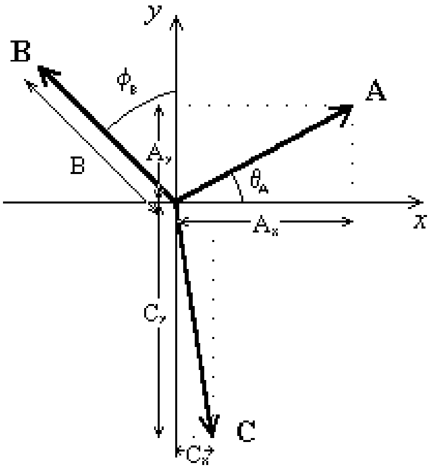
or

$$C_x = -A\cos\theta_A + B\sin\phi_B$$

and

$$C_y = -A\sin\theta_A - B\cos\phi_B\,.$$

These last two equations give us both components for **C** and these values enable us to draw **C** in the following diagram.



The length of **C** (or $|\mathbf{C}|$) is found from

$$C = \sqrt{C_x^2 + C_y^2}$$

while

$$\theta_C = \arctan\left(\frac{C_y}{C_x}\right)$$

and in our example we would expect this angle to lie in the fourth quadrant.

Another common case that requires zero sum vectors is the application of the *principle of conservation of momentum*, when two particles collide the sum of their final momenta must be equal to the sum of their initial momenta. Symbolically this is written as

$$\mathbf{p}_{ai} + \mathbf{p}_{bi} = \mathbf{p}_{af} + \mathbf{p}_{bf}$$

To find the final momentum of particle b that is $\mathbf{p}_{bf}$ we would use

$$(p_{bf})_x = (p_{ai})_x + (p_{bi})_x - (p_{af})_x$$

and

$$(p_{bf})_y = (p_{ai})_y + (p_{bi})_y - (p_{af})_y..$$

For both examples of vector sums given in this section you are not expected to learn formulae, rather you are expected to take numerical examples and solve them by applying the techniques outlined in this section.

2.5 Relative quantities

In all measurements that you make the result is a quantity that is given with reference to another quantity; for example, if the speedometer of your car reads 60 km hr^{-1} you may not think that you are speeding, but if the "speedo" doesn't start to register until the car is moving at 30 km hr^{-1} then you should not trust subsequent readings. Again the statement that "I have been here for two hours." implies that two times were measured , one at the beginning and the other when the calculation was made.

The rule for finding quantity B relative to quantity A, is to subtract the value for A from B. If $\mathbf{s}_A$ is the displacement of A and $\mathbf{s}_B$ is the displacement of B, then the displacement of B *relative* to A is

$$\mathbf{s}_{BA} = \mathbf{s}_B - \mathbf{s}_A \ .$$

Very often, a given time interval is written as

t instead of $t - t_0$

where the time is relative to $t_0 = 0$ (perhaps the time when a stopwatch was started).