CHAPTER 1.

MEASUREMENT AND UNCERTAINTY

1.1 Introduction

Ultimately all Physical theories must be verified by experimental measurement, for this reason we will start this physics course by considering the way in which we may relate measurements with theory. An *accurate* observation is one in which we are completely confident, with access to all available knowledge, that it is correct. For instance it is accurate to say that we are kept against the surface of the earth because of the force of gravity. For a quantitative description we can also say that the acceleration of an object falling freely to earth is 9.8 m s^{-2}, this number is accurate although you may later find that there are many variations and exceptions to this value. A word that can be closely related to accuracy is *precision*. This word indicates the amount of uncertainty that might be associated with our description of a measurement. From our first example increasing the precision to say that the acceleration of a falling object is 9.800 m s^{-2} would be more precise, but make the value unreliable and hence inaccurate. An accurate clock is one that does not need correcting, nevertheless in using such a clock we make readings with limited precision; for example if someone asks for the time from an accurate clock they will be happy knowing that it is say, four minutes past three; although the clock can be read with greater precision it is a little unusual to give the time to the nearest second. In a similar fashion when you are asked to give your age you usually answer with a precision of one year, this is the case even if your age is needed for legal reasons or a scientific survey.

1.2 Significant figures

In normal practise three significant figures meet the needs of most descriptions in physics.
Some examples of three significant figures are: 34900, 0.00123, 24.1.
Some examples that are not using three significant figures are: 34 900.0, 34 901, 1.001 23, 24.10.

For the examples given to three significant figures there is an implied uncertainty. The number 34 900 may be taken to be less than 34 950 and greater than 34 850 this uncertainty leads us to assume that:

$$34\,900 \Rightarrow 34\,900 \pm 50$$
$$0.001\,23 \Rightarrow 0.001\,23 \pm 0.000\,005$$
$$24.1 \Rightarrow 24.1 \pm 0.05.$$

There are often ambiguities where common sense is needed. One case can be found above with the number 34 900, what if the crowd at a football match was very carefully counted and was found to be exactly this number? If you heard the figure over the radio in a match report, you would never know how carefully it was measured and probably assume that the number was accurate with the implied uncertainty of ± 50. As another example, the number 12.49 may be rounded up to 12.5 and then 13, or it may be rounded directly down to 12, thus either 12 or 13 could represent 12.49 to two significant figures.

People often confuse decimal places with significant figures; 1.243 has four significant figures and three decimal places. This confusion is overcome by expressing a quantity in scientific notation.

$$34\,900 = 3.49\times10^4,$$
$$0.001\,23 = 1.23\times10^{-3}$$
and
$$24.1 = 2.41\times10.$$

In each case we have three significant figures and two decimal places; in the last case it would be best to round off as desired and then return to 24.1.

1.3 Uncertainties

In normal laboratory measurements as well as every day discussions, we are content to work to about 1% precision. For example we may or may not bother to offer $99 for a $100 item. Again we may be content to convert the old standard measurement of 1 foot to 30 cm, even though 1.000 feet = 304.8 mm is the exact conversion.

We have already used the form "34 900 ± 50" to indicate the uncertainty in a number. The significant figures in the uncertainty and those in the quantity should agree to the same order. For example the following appear to be correct: 1.23 ± 0.05, $2.46\times10^3 \pm 20$ and 0.0024 ± 0.0001. By contrast next set appear to be incorrect: 1.23 ± 0.052, $2.46\times10^3 \pm 2$ and 0.0024 ± 0.00001. This last set would be better written as: 1.230 ± 0.052, $2.460\times10^3 \pm 2$ and 0.00240 ± 0.00001.

Just as we can't set absolute rules for using significant figures, we have similar problems when presenting uncertainties or possible errors. As an example a number of measurements of a particular quantity may be presented as 32.41 ± 0.63, if this is taken to be correctly presented then the statement implies that a large number of determinations and a careful statistical analysis was done before presenting this result; the presenter must have taken great pains to carefully establish the range of uncertainty.

When quantities with given uncertainties are combined by a mathematical operation, the outcome must have a corresponding uncertainty. Form the following examples we will suppose that $a = 2.43 \pm 0.02$ and $b = 1.04 \pm 0.01$.

Addition Consider $a + b$.
We use the maximum value of a (2.45) and add this to the maximum value of b (1.05) to get the maximum value of $a + b = 2.45 + 1.05 = 3.50$.
Then we do the same with the minimum values to get the minimum value of

$a + b = 2.41 + 1.03 = 3.44.$

The desired result must lie in the range 3.44 to 3.50, we can quote our answer as the average value of the minimum and maximum, $a + b = \frac{3.50 + 3.44}{2} = 2.47$ while half the range gives the appropriate uncertainty, $\frac{3.50 - 3.44}{2} = 0.03$.

Our desired result should be $a + b = 3.47 \pm 0.03.$

In this case $a+b$ with its uncertainty will include the extreme values.

Subtraction Consider $a - b$.

For the difference $a - b$ the procedure is much the same as before, we find the minimum and maximum values and quote the average of these, with half the range as the uncertainty.

The maximum of $a - b = 2.45 - 1.03 = 1.42$,

the minimum of $a - b = 2.41 - 1.05 = 1.36$,

so that the required result is $a - b = 2.39 \pm 0.03$.

You will need to look carefully at the way in which the maximum or minimum values of a and b have been combined to give the extreme results for this difference. The overriding rule for these determinations is to first think carefully how you can maximise and minimise the result, this correct procedure is not always obvious.

Multiplication Consider $a\times b$.

In the case of a product the procedure is still straightforward:

the maximum of $a\times b = 2.45\times 1.05 = 2.5725$,

the minimum of $a\times b = 2.41\times 1.03 = 2.4823$,

while this implies $a \times b = 2.5274 \pm 0.0451$

the required result is $a\times b = 2.53 \pm 0.05$.

In this example we allowed the calculation to extend out to five significant figures because this working is not the required answer. We should quote our answer with a precision that does not exceed the precision of the original factors (a and b were originally given to three significant figures). Notice also that the maximum implied in the result is 2.58 which is considerably higher than the rounded value of 2.5725, this choice is a matter of judgement, we could have chosen 2.57 instead. As we are dealing with uncertainties we could also claim that it is unlikely that both extreme values would occur together so that

$$a\times b = 2.53 \pm 0.04$$

could also be considered as an acceptable answer.

Division Consider a/b

In the case of division we need to carefully maximise and minimise the result. We maximise this result by dividing the maximum value of a by the minimum value of b, while we minimise the result by dividing the minimum value of a by the maximum value of b.

The maximum of $a/b = 2.45/1.03 = 2.379$

the minimum of $a/b = 2.41/1.05 = 2.295$

so that $a/b = 2.34 \pm 0.04$,

is the acceptable answer. An argument could also be made in favour of 2.34 ± 0.05.

1.4 The mean and deviation

If a single quantity x is determined (perhaps in an experiment) a number of times and different values are obtained, then we are often content to quote the average value as the final result. To find this average each value can be distinguished by a subscript j as x_j, for example we could take the data set:

$\{x_1 = 12.1,\ x_2 = 12.9,\ x_3 = 12.5,\ x_4 = 12.1,\ x_5 = 12.3,\ x_6 = 12.8,\ x_7 = 12.7,\ x_8 = 12.5\}$.

The mean of these values is calculated using

$$\bar{x} = \frac{\sum_{j=1}^{n} x_j}{n}, \qquad (1.1)$$

or $$\bar{x} = \frac{12.1 + 12.9 + 12.5 + 12.1 + 12.3 + 12.8 + 12.7 + 12.5}{8}$$

so that $$\bar{x} = 12.49.$$

A quick inspection of the range covered by the original data set $\{x_j\}$will tell us that our value for x should be

$$\bar{x} = 12.5 \pm 0.4.$$

From a statistical point of view, it would be less likely that one more measurement would give us 12.9 or 12.1, the extreme values so for purposes of prediction of further measurements we might chose to reduce the range of uncertainty a little. There are several formulae that enable estimate a reduced uncertainty, the most common one is that for the standard deviation,

$$\sigma = \sqrt{\frac{\sum_{j=1}^{n}\left(x_j - \bar{x}\right)^2}{n}}.$$

Although this formula can be a bit of a nuisance to use, most student calculators are pre-programmed to calculate this (or a similar) standard deviation. When we apply this formula to our data set, along with the mean value , we get standard deviation of

$$\sigma = 0.285 .$$

This time, by using formulae we can also quote our value of x as

$$x = 12.5 \pm 0.3 .$$

In this case the use of formulae appears to have improved our uncertainty from 0.4 to 0.3. Statistical formulae do not magically improve results, for small data sets they should merely reflect the obvious. Many student scientists seem to get a false sense of satisfaction from minimising the uncertainties in their measurements, they use formulae and other means to suggest that their results have an unwarranted precision and hence accuracy.

1.5 Units

Many physical quantities are meaningless without the inclusion of the correct units. For example: the statement "You will have to travel three to get home." has no meaning until you learn that the units are kilometres.

The primary units that will initially concern you are:

i time; *seconds*, symbol " s ", the commonly used variable is t ,

ii length; *metre*, symbol " m ", the commonly used variables are d, l, s, or r,

iii mass; *kilogram*, symbol " kg ", the commonly used variable is m ,

iv electric current; *ampere*, symbol " A ", the commonly used variable is i or I.

There are also derived units that you must learn to use, for example the units of force are *newtons* (symbol " N "). One newton is the force required to cause a mass of 1 kg to accelerate at the rate of 1 m s^{-2} , or stated mathematically

$$1 \text{ N} = 1 \text{ kg m s}^{-2}.$$

Many textbooks still include units in the following style

$$1 \text{ N} = 1 \text{ kg m / s}^{2},$$

this style can lead to ambiguities and will not be used in this course.

We have already recommended the use of scientific notation as a useful way of expressing numbers and this also applies to units. The standard prefixes are:

i pico; symbol " p ", representing 10^{-12},

ii nano; symbol " n ", representing 10^{-9},

iii micro; symbol " μ ", representing 10^{-6},

iv milli; symbol " m ", representing 10^{-3},

v kilo; symbol " k ", representing 10^{3},

vi mega; symbol " M ", representing 10^{6},

vii giga; symbol " G ", representing 10^{9}.

The prefix centi is still in use representing 10^{-2}, but is no longer official. This system, using powers of ten that are multiples of three, is sometimes called "engineering notation". Some other, not quite official, units are still commonly used; one is the litre (10^{-3} m^{3}), another is the calorie (4.19 kg m^{2} s^{-2} or J). You may also have problems understanding the approximate values of some of the older, now obsolete, units such as "stone", "inches", and "acres".

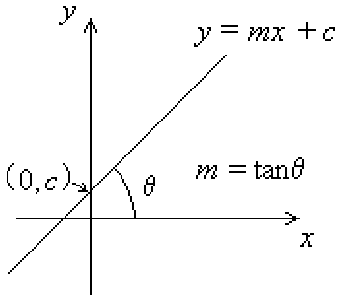
1.6 Graphical representation

You should be familiar with equations of the type

$$y = mx + c,$$

these give a straight line relationship between the vertical ordinates (or y values) and the horizontal abscissae (or the x values). This relation between x and y is often called a linear relationship, linear functions have a rather different meaning that you may encounter in later mathematics.

The straight line looks like this:

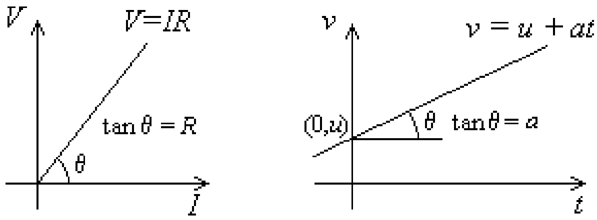


It takes some time before all students can recognise that equations like

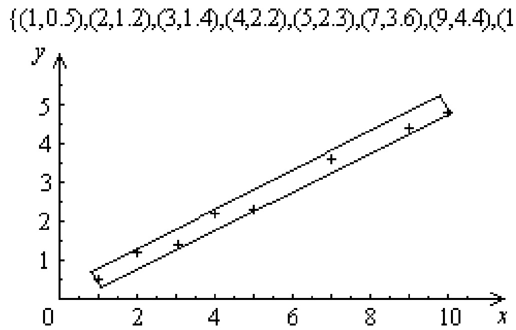
$$V = IR$$

or $$v = u + at$$

are also straight line relations, (R is the slope of the V against I graph) or (a is the slope of the v against t graph and u is the intercept).



Now we take a data set generated from a linear relationship, but with some uncertainties added $\{(x,y)\} = \{(1,0.5),(2,1.2),(3,1.4),(4,2.2),(5,2.3),(7,3.6),(9,4.4),(10,4.8)\}$
and plot these on a graph.



As we have added uncertainties for each point we can no longer tell what the original relation was. In order to estimate the range of straight lines that might fit this data set the co-ordinates have been enclosed by a quadrilateral (nearly a rectangle). The corners of this quadrilateral are at (0.8,0.7), (9.8,5.25), (1.05,0.3) and (10.15, 4.8). Two straight lines can be drawn as diagonals through the corners of the quadrilateral, either of these lines will appear as a good data fit. We can then use these corners to get the slopes of the lines and the intercepts. To do this we can apply the formula

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

to these diagonals this will give:

$$y = 0.44x + 0.35$$

and

$$y = 0.57x - 0.29.$$

From these values we can say that the slope of the line is

$$m = 0.51 \pm 0.07$$

and the intercept is $c = 0.03 \pm 0.32$.

You should not get too concerned about the uncertainty being greater than the actual value of c.

1.7 Estimations

A common fault with student answers, and one that we have been trying to correct, is to give a numerical answer to too many significant figures. This fault indicates that the student has not stopped to reflect on the meaning of the number that has been given. We will run through some estimates using just one or two significant figures so that can obtain a result with the precision of an *order of magnitude*. An answer such as 3×10^3 is correct to an order of magnitude when 10^4 is considered to be the maximum uncertainty and 10^3 is the minimum uncertainty. In this section we are not demonstrating how to calculate uncertainties, we are trying to show you how to test whether or not a result is sensible or "about right".

Density is the mass of a body divided by its volume:

$$\rho = \frac{m}{V}, \qquad (1.2)$$

where m is mass and V is the volume. According to the old standard of mass 1×10^{-6} m^3 (or 1 cm^3) of water had a mass of 1×10^{-3} kg (or 1 gm), hence the density of water is

$$\rho = \frac{10^{-3}\ \mathrm{kg}}{10^{-6}\ \mathrm{m}^3} = 10^3\ \mathrm{kg\ m^{-3}}.$$

This result is correct to three or four places because it involves standard values.
According to the observations of Archimedes, any object with a greater density than water will sink in the water while any object with a lower density than that of the water will float on the water. People just float in water (until their lungs are empty or full of water) and so we can assume that people also have a density of 1000 kg m^3. The volume of a 60 kg person can be calculated using the density of water

$$V = \frac{m}{\rho} = \frac{60\ \mathrm{kg}}{1000\ \mathrm{kg\ m^{-3}}} = 0.06\ \mathrm{m}^3.$$

It is an extremely difficult business to calculate or measure the volume of a person any other way, the method of estimation gives a simple and reliable means. Now people are made of carbon, hydrogen and oxygen with most other atoms in small quantities, as we are estimating we will assume the average atomic mass of a person to be 10, this means that we expect

Avogadro's number 6×10^{23} atoms to have a mass of 10×10^{-3} kg. From this we can estimate that each atom will have a mass of

$$m_{atom} = \frac{10^{-2}\ kg}{6\times10^{23}} = 1.7\times10^{-26}\ kg.$$

As an estimate this value is indeed to found to be about right. If one atomic mass is 10×10^{-3} kg, our 60 kg person must have $\frac{60\ kg}{10^{-2}\ kg} = 6000$ average atomic masses each with or 6×10^{23} atoms. If this person has N atoms then

$$N = 6000\times6\times10^{23} = 36\times10^{26} \approx 4\times10^{27}$$

or they contain 4×10^{27} atoms (for purposes of approximation 3.6≈4). The volume for each atom in our estimate is $V_{atom} = \frac{0.6\ m^3}{4\times10^{27}} = 1.5\times10^{-29}\ m^3$; now if each atom occupies a cube, the length of a side of that cube will be the cube root of 10^{-29}, that is $(10^{-29})^{1/3}$. Rather than use a calculator to calculate an rough estimate, we can call this last number $(8\times10^{-30})^{1/3}$, so that the cube root is seen to be 2×10^{-10}. Our spacing between atoms is about this distance and the figure is known to be correct to within an order of magnitude.

If you cannot follow the details of this argument yet, you should at least follow the mathematics of the estimate.